MAT124 MATHEMATICS II

The Divergence Theorem and Stokes's Theorem

Outline

The Divergence Theorem

Stokes's Theorem

The Divergence Theorem

Green's Theorem in the Plane

The Two-Dimensional Divergence Theorem

The following theorem is an alternative formulation of the two-dimensional Fundamental Theorem of Calculus. In this case we express the double integral of $\operatorname{div} \mathbf{F}$ (a derivative of $\mathbf{F}$) over R as a single integral of the outward normal component of $\mathbf{F}$ on the boundary $\mathcal{C}$ of R.

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The Divergence Theorem in the Plane

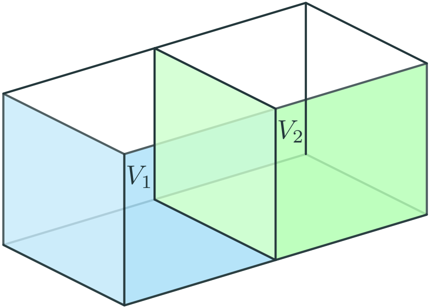
Let R be a regular, closed region in the xy-plane whose boundary, $\mathcal{C}$, consists of one or more piecewise smooth, simple closed curves. Let $\hat{\mathbf{N}}$ denote the unit outward (from R) normal field on $\mathcal{C}$. If $\mathbf{F} = F_1(x,y)\mathbf{i} + F_2(x,y)\mathbf{j}$ is a smooth vector field on R, then

$$\iint_R \operatorname{div}\mathbf{F}\,dA = \oint_{\mathcal{C}} \mathbf{F}\cdot\hat{\mathbf{N}}\,ds.$$

The Divergence Theorem in 3-Space

The Cancellation Principle

Recall that **divergence** measures how much fluid or flux is expanding *out* of a tiny volume. What happens if we place two tiny boxes next to each other?



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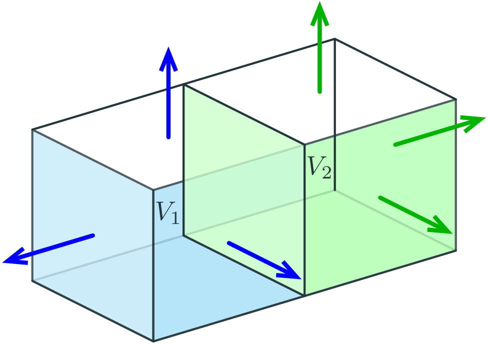

If we sum the total outward flux of both boxes:

$$\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS + \iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS$$

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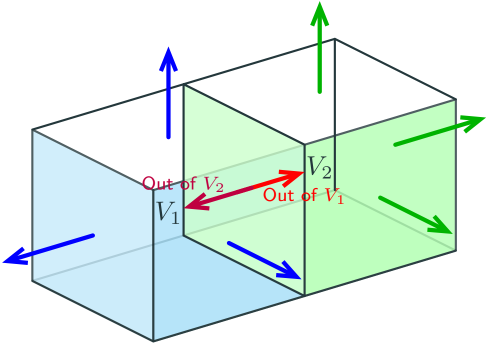

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Notice the shared wall. The flux leaving V_1 goes **RIGHT**, while the flux leaving V_2 goes **LEFT**.

Whatever flows *out* of V_1 exactly flows *into* V_2. They perfectly **cancel each other out!**

The Divergence Theorem in 3-Space

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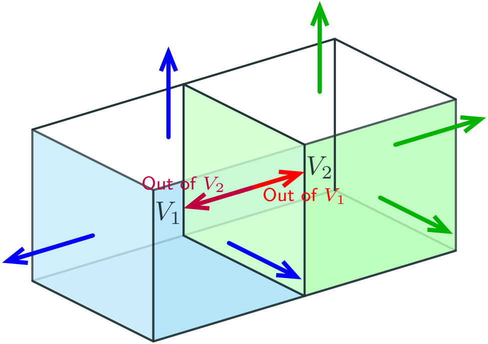

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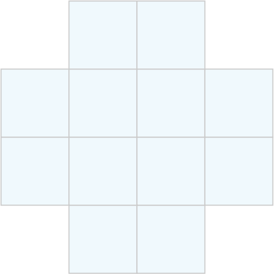
The net result is simply the flux through the **outer boundary** of the combined shape.

The Divergence Theorem

From Microscopic to Macroscopic

Imagine dividing a large macroscopic 3D region D into a grid of tiny boxes.



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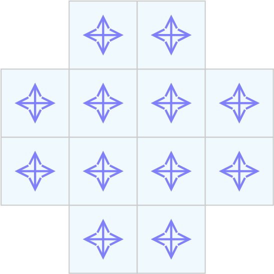

The outward flux of a single tiny box i is its local divergence multiplied by its tiny volume:

$$\iint_{S_i} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS \approx (\operatorname{div} \mathbf{F})_i\, \Delta V$$

The Divergence Theorem

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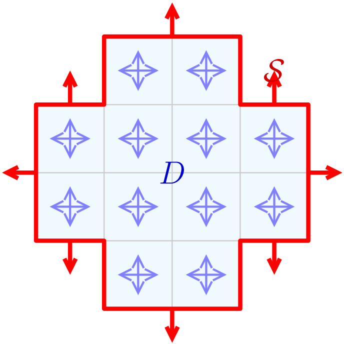

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$$\iint_{S_i} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS \approx (\operatorname{div} \mathbf{F})_i\, \Delta V$$

If we sum the flux of all tiny boxes, **all internal flows cancel out** because they cross shared walls in opposite directions. Only the flux through the outer boundary $\mathcal{S}$ has no neighbor to cancel with!

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV$$

The Divergence Theorem in 3-Space

We say the three-dimensional domain D is x-simple if it is bounded by a piecewise smooth surface $\mathcal{S}$ and if every straight line parallel to the x-axis and passing through an interior point of D meets $\mathcal{S}$ at exactly two points.

Similar definitions hold for y-simple and z-simple, and we call the domain D **regular** if it is a union of finitely many, nonoverlapping subdomains, each of which is x-simple, y-simple, and z-simple.

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The Divergence Theorem (Gauss's Theorem)

Let D be a regular, three-dimensional domain whose boundary $\mathcal{S}$ is an oriented, closed surface with unit normal field $\hat{\mathbf{N}}$ pointing out of D. If $\mathbf{F}$ is a smooth vector field defined on D, then

$$\iiint_D \operatorname{div} \mathbf{F}\, dV = \oiint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS.$$

The Divergence Theorem in 3-Space

The Divergence Theorem can be used in both directions to **simplify explicit calculations of surface integrals** or **volumes**.

We give examples of each.

The Divergence Theorem in 3-Space

EXAMPLE

Let $\mathbf{F} = bxy^2\mathbf{i} + bx^2y\mathbf{j} + (x^2 + y^2)z^2\mathbf{k}$, and let $\mathcal{S}$ be the closed surface bounding the solid cylinder R defined by $x^2 + y^2 \leq a^2$ and $0 \leq z \leq b$.

Find $\oiint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

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Solution: By the Divergence Theorem, $\displaystyle\oiint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iiint_R \operatorname{div}\mathbf{F}\,dV$.

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$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial}{\partial x}(bxy^2) + \frac{\partial}{\partial y}(bx^2y) + \frac{\partial}{\partial z}\left((x^2+y^2)z^2\right)\\ &= by^2 + bx^2 + 2z(x^2+y^2) = (x^2+y^2)(b+2z)\end{aligned}$$

The Divergence Theorem in 3-Space

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So, converting to cylindrical coordinates ($x^2+y^2=r^2$, $dV = r\,dr\,d\theta\,dz$):

$$\begin{aligned} \iiint_R \operatorname{div} \mathbf{F}\, dV &= \int_0^b (b+2z)\,dz \int_0^{2\pi} d\theta \int_0^a r^2(r\,dr) \\ &= (b^2+b^2)(2\pi)\left(\frac{a^4}{4}\right) = \pi a^4 b^2. \end{aligned}$$

The Divergence Theorem in 3-Space

EXAMPLE

Evaluate $\oiint_{\mathcal{S}} (x^2 + y^2)\, dS$, where $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = a^2$. Use the Divergence Theorem.

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$$\hat{\mathbf{N}} = \frac{\mathbf{r}}{a} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}.$$

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We would like to choose $\mathbf{F}$ so that $\mathbf{F} \cdot \hat{\mathbf{N}} = x^2 + y^2$. Observe that $\mathbf{F} = a(x\mathbf{i} + y\mathbf{j})$ will do:

$$\mathbf{F} \cdot \hat{\mathbf{N}} = a(x\mathbf{i} + y\mathbf{j}) \cdot \left(\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} \right) = x^2 + y^2.$$

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If B is the solid ball bounded by $\mathcal{S}$, then $\operatorname{div} \mathbf{F} = a(1) + a(1) + 0 = 2a$, so:

$$\begin{aligned} \oiint_{\mathcal{S}} (x^2 + y^2)\, dS &= \oiint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = \iiint_B \operatorname{div} \mathbf{F}\, dV \\ &= \iiint_B 2a\, dV = (2a)\left(\frac{4}{3}\pi a^3 \right) = \frac{8}{3}\pi a^4. \end{aligned}$$

The Divergence Theorem in 3-Space

EXAMPLE

By using the Divergence Theorem with $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, calculate the volume of a cone having base area A and height h. The base can be any smoothly bounded plane region.

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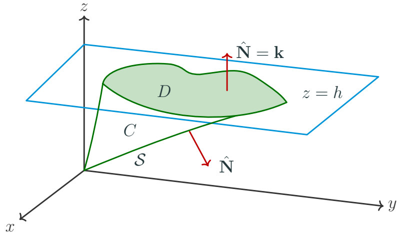
Solution: Let the vertex of the cone be at the origin and the base in the plane $z = h$. The solid cone $\mathcal{C}$ has surface consisting of two parts: the conical wall $\mathcal{S}$ and the base region D that has area A.



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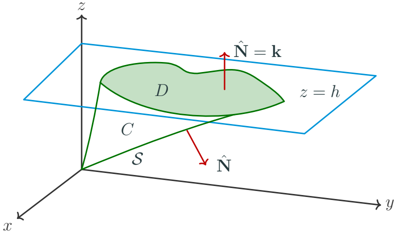
Since $\mathbf{F}(x, y, z)$ points directly away from the origin at any point $(x, y, z) \neq (0, 0, 0)$, we have $\mathbf{F} \cdot \hat{\mathbf{N}} = 0$ on the walls $\mathcal{S}$. On D, we have $\hat{\mathbf{N}} = \mathbf{k}$ and $z = h$, so $\mathbf{F} \cdot \hat{\mathbf{N}} = z = h$ on the base.



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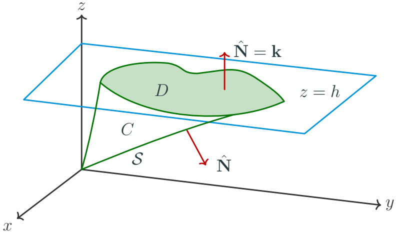

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Since $\operatorname{div}\mathbf{F}(x, y, z) = 1 + 1 + 1 = 3$, we have, by the Divergence Theorem,

$$3V = \iiint_{\mathcal{C}} \operatorname{div}\mathbf{F}\, dV = \iint_{\mathcal{S}} \mathbf{F}\cdot\hat{\mathbf{N}}\, dS + \iint_{D} \mathbf{F}\cdot\hat{\mathbf{N}}\, dS = 0 + h\iint_{D} dS = Ah \Rightarrow \boxed{V = \frac{Ah}{3}}$$

The Divergence Theorem in 3-Space

EXAMPLE

Find the flux of $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$ upward through the first octant part $\mathcal{S}$ of the cylindrical surface $x^2 + z^2 = a^2$, $0 \leq y \leq b$.

Solution:

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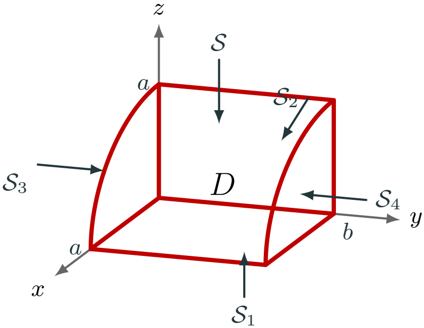

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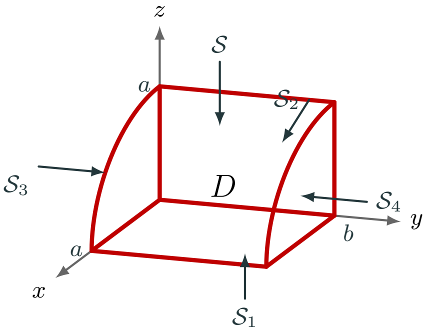

$\mathcal{S}$ is one of five surfaces that form the boundary of the solid region D. The other four surfaces are planar. Orient all these surfaces with normal $\hat{\mathbf{N}}$ pointing out of D. Then

- $\mathcal{S}_1$ lies in $z = 0$ $(\hat{\mathbf{N}} = -\mathbf{k})$ $\implies \mathbf{F} \cdot \hat{\mathbf{N}} = -z = 0.$
- $\mathcal{S}_2$ lies in $x = 0$ $(\hat{\mathbf{N}} = -\mathbf{i})$ $\implies \mathbf{F} \cdot \hat{\mathbf{N}} = -x = 0.$
- $\mathcal{S}_3$ lies in $y = 0$ $(\hat{\mathbf{N}} = -\mathbf{j})$ $\implies \mathbf{F} \cdot \hat{\mathbf{N}} = -y^2 = 0.$
- $\mathcal{S}_4$ lies in $y = b$ $(\hat{\mathbf{N}} = \mathbf{j})$ $\implies \mathbf{F} \cdot \hat{\mathbf{N}} = y^2 = b^2.$

The Divergence Theorem in 3-Space

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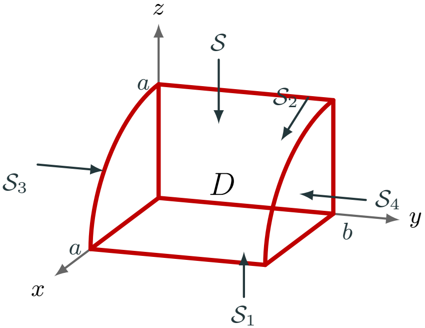
Solution:



If $\mathcal{S}_{tot}$ denotes the whole boundary of D, then:

$$\oiint_{\mathcal{S}_{tot}} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = \iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS + 0 + 0 + 0$$

$$+ \iint_{\mathcal{S}_4} b^2\, dS$$

$$= \iint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS + b^2 \left(\frac{\pi a^2}{4} \right).$$

The Divergence Theorem in 3-Space

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Find the flux of $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + z\mathbf{k}$ upward through the first octant part $\mathcal{S}$ of the cylindrical surface $x^2 + z^2 = a^2$, $0 \le y \le b$.

Solution (continued): On the other hand, by the Divergence Theorem,

$$\iint_{\mathcal{S}_{tot}} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV = \iiint_D (1 + 2y + 1)\, dV = \iiint_D (2 + 2y)\, dV.$$

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This integral can be evaluated directly or using centroid properties:

$$\iiint_D (2 + 2y)\, dV = 2V + 2V\bar{y},$$

where $V = \frac{\pi a^2 b}{4}$ is the volume of D, and $\bar{y} = \frac{b}{2}$ is the y-coordinate of the centroid of D.

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Combining these results, the flux of $\mathbf{F}$ upward through $\mathcal{S}$ is:

$$\oiint_{\mathcal{S}} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = 2V\,(1 + \bar{y}) - \frac{\pi a^2 b^2}{4} = 2\left(\frac{\pi a^2 b}{4}\right)\left(1 + \frac{b}{2}\right) - \frac{\pi a^2 b^2}{4} = \frac{\pi a^2 b}{2}.$$

The Divergence Theorem in 3-Space

Variants of the Divergence Theorem

Other versions of the Fundamental Theorem of Calculus can be derived from the Divergence Theorem. Two are given in the following theorem:

The Divergence Theorem in 3-Space

Variants of the Divergence Theorem

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THEOREM

If D satisfies the conditions of the Divergence Theorem and has surface $\mathcal{S}$, and if $\mathbf{F}$ is a smooth vector field and ϕ is a smooth scalar field, then

(a)

$$\iiint_D \operatorname{curl} \mathbf{F}\, dV = -\oiint_{\mathcal{S}} \mathbf{F} \times \hat{\mathbf{N}}\, dS,$$

(b)

$$\iiint_D \operatorname{grad} \phi\, dV = \oiint_{\mathcal{S}} \phi \hat{\mathbf{N}}\, dS.$$

Stokes's Theorem

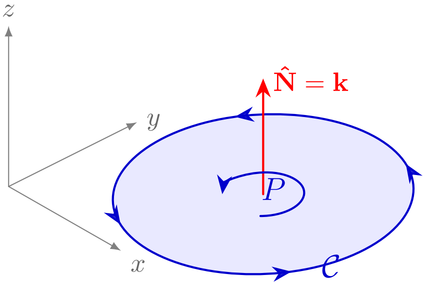
Stokes's Theorem

From Green's Theorem to 3D Space

If we regard a region R in the xy-plane as a surface in 3-space with normal field $\hat{\mathbf{N}} = \mathbf{k}$, then the Green's Theorem formula can be written in the form:

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{R} \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{N}} \, dS$$

where $\mathcal{C}$ is the boundary of R with orientation implied by the normal field.



The Leap to 3D

Stokes's Theorem extends this exact result to more general surfaces that are **nonplanar** (curved).

Stokes's Theorem

Stokes's Theorem

Let $\mathcal{S}$ be a piecewise smooth, oriented surface in 3-space, having unit normal field $\hat{\mathbf{N}}$ and boundary $\mathcal{C}$ consisting of one or more piecewise smooth, closed curves with orientation inherited from $\mathcal{S}$.

If $\mathbf{F}$ is a smooth vector field defined on an open set containing $\mathcal{S}$, then

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS.$$

Stokes's Theorem

EXAMPLE

Evaluate $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k}$, and $\mathcal{C}$ is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $2x + 2y + z = 3$ oriented so as to have a counterclockwise projection onto the xy-plane.

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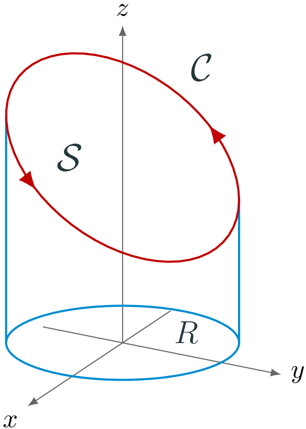

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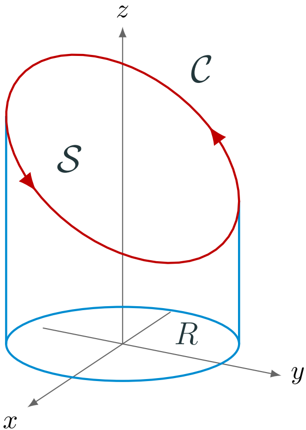

$\mathcal{C}$ is the oriented boundary of an elliptic disk $\mathcal{S}$ that lies in the plane $2x + 2y + z = 3$ and has the circular disk R: $x^2 + y^2 \leq 1$ as projection onto the xy-plane.

On $\mathcal{S}$ we have $\hat{\mathbf{N}}\, dS = (2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\, dx\, dy$.

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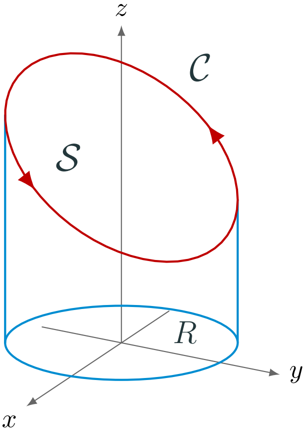

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Also,

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = 3(x^2 + y^2)\mathbf{k}.$$

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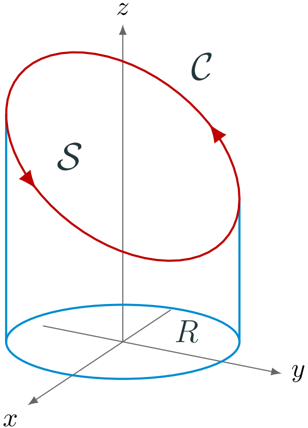

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On $\mathcal{S}$ we have $\hat{\mathbf{N}}\, dS = (2\mathbf{i} + 2\mathbf{j} + \mathbf{k})\, dx\, dy$. Thus, by Stokes's Theorem,

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS \\ &= \iint_{R} 3(x^2 + y^2)\, dx\, dy \\ &= 2\pi \int_0^1 3r^2 (r\, dr) = \frac{3\pi}{2}.\end{aligned}$$

Stokes's Theorem

EXAMPLE

Find $I = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS$, where $\mathcal{S}$ is that part of the sphere $x^2 + y^2 + (z-2)^2 = 8$ that lies above the xy-plane, $\hat{\mathbf{N}}$ is the unit outward normal field on $\mathcal{S}$, and $\mathbf{F} = y^2 \cos(xz)\mathbf{i} + x^3 e^{yz}\mathbf{j} - e^{-xyz}\mathbf{k}$.

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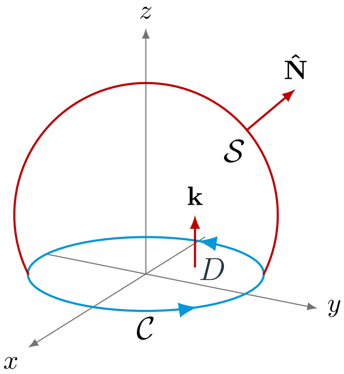

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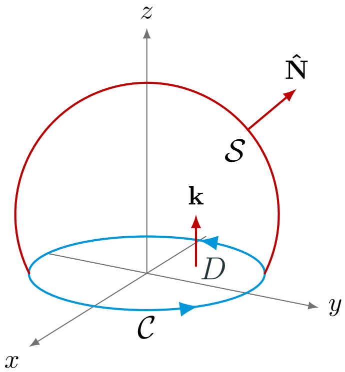

The boundary, $\mathcal{C}$, of $\mathcal{S}$ is the circle $x^2 + y^2 = 4$ in the xy-plane, oriented counterclockwise as seen from the positive z-axis.

This curve is also the oriented boundary of the plane disk D: $x^2 + y^2 \le 4$, $z = 0$, with normal field $\hat{\mathbf{N}} = \mathbf{k}$. Thus, two applications of Stokes's Theorem give:

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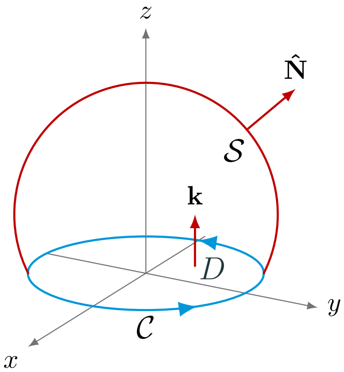

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$$I = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot \hat{\mathbf{N}}\, dS = \oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{D} \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA.$$

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Solution (continued): On D $(z = 0)$ we have:

$$\begin{aligned} \operatorname{curl} \mathbf{F} \cdot \mathbf{k} &= \left(\frac{\partial}{\partial x}(x^3 e^{yz}) - \frac{\partial}{\partial y}(y^2 \cos(xz)) \right) \Bigg|_{z=0} \\ &= 3x^2 - 2y. \end{aligned}$$

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So the integral simplifies to:

$$I = 3 \iint_D x^2\, dA = 3 \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^2 r^3\, dr = 3(\pi) \left[\frac{r^4}{4} \right]_0^2 = 3(\pi)(4) = 12\pi.$$

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When Stokes's Theorem Fails

Remark

A surface $\mathcal{S}$ satisfying the conditions of Stokes's Theorem may no longer do so if a single point is removed from it. An isolated boundary point of a surface is not an orientable curve, and Stokes's Theorem may therefore break down for such a surface.

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Consider, for example, the vector field

$$\mathbf{F} = \frac{\hat{\boldsymbol{\theta}}}{r} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j},$$

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If D is oriented with upward normal $\mathbf{k}$, then its boundary consists of the oriented, smooth, closed curve, $\mathcal{C}$, given by
$x = a\cos\theta, y = a\sin\theta, (0 \leq \theta \leq 2\pi)$, and the isolated point $(0,0)$. We have:

$$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \left(\frac{-\sin\theta}{a}\mathbf{i} + \frac{\cos\theta}{a}\mathbf{j}\right)\cdot(-a\sin\theta\mathbf{i} + a\cos\theta\mathbf{j})\,d\theta = 2\pi.$$

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However,

$$\operatorname{curl}\mathbf{F} = \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right]\mathbf{k} = \mathbf{0}$$

identically on D. Thus,

$$\iint_D \operatorname{curl}\mathbf{F}\cdot\hat{\mathbf{N}}\,dS = 0,$$

and the conclusion of Stokes's Theorem fails in this case!